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Mean-field renormalization group for the boundary magnetization of strip clusters

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Abstract. We analyse in some detail a recently proposed transfer matrix mean-field approximation which yields the exact critical point for several two-dimensional nearest-neighbour Ising models. For the square lattice model we show explicitly that this approximation yields not only the exact critical point, but also the exact boundary magnetization of a semi-infinite Ising model, independent of the size of the strips used. Then we develop a new mean-field renormalization-group strategy based on this approximation and make connections with finite size scaling. Applying our strategy to the quadratic Ising and three-state Potts models we obtain results for the critical exponents which are in very good agreement with the exact ones. In this way we also clarify some advantages and limitations of the mean field renormalization group approach.

1. Introduction

A recently proposed [1] transfer matrix version of a mean-field approximation (which in the following will be denoted by LS) applied to several nearest neighbour Ising models in two dimensions, gave surprisingly exact results for the critical points, even without extrapolation, and very good results, under extrapolation, for more complicated models.

A first issue to address in connection with the LS approximation is why the results are exact in the NN Ising case and extrapolate accurately in the others, and whether extra exact results can be obtained by this scheme.

We want also to clarify what is the connection of this method with other techniques of more common use in two-dimensional statistical mechanics. In particular:

(i) since the method involves consideration of strips similar to those used, e.g. in finite size scaling (FSS) and phenomenological renormalization approaches [2], it is legitimate to ask up to which extent the LS approximation is connected to these approaches and possibly fits within them;

(ii) since the method uses as a basic ingredient effective fields on the boundary of the strips, it is also rather natural to look for connections with the so-called mean-field renormalization-group [3] (MFRG) approach. This will go together with showing how critical exponents can be obtained in this context.

The plan of the paper is as follows: in section 2 we give a brief review of the LS approximation and show, in the square lattice Ising case, that it gives not only the exact critical point, but also the exact boundary magnetization of a semi-infinite Ising model, independent of the size of the strips used; in section 3 we show how the LS approximation fits into a MFRG structure, develop a procedure for calculating the critical exponents, and compare the method with the FSS approach; in section 4 we give two test applications of the procedure above, on the two-dimensional Ising and three-state Potts models. Finally, in section 5, we draw some conclusions.

2. The LS approximation

The LS approximation scheme [1] makes use of two infinite strips $\mathcal{S}_n$ and $\mathcal{S}_{n'}$ of widths n and n' respectively, with periodic boundary conditions along the infinite direction. The approximation is obtained by applying an effective field h_{eff} at *one side* of the strips and by imposing the consistency relation

$$m_{1n}(K, h_{\text{eff}}) = m_{1n'}(K, h_{\text{eff}}) \tag{1}$$

where m_{1n} and $m_{1n'}$ are the values of the order parameter at *the opposite side* of the strips and $K = \beta J > 0$ is the exchange interaction strength. Equation (1) has to be solved for h_{eff} at fixed K and the critical temperature is the one for which the paramagnetic solution $h_{\text{eff}} = 0$ bifurcates into non-zero solutions, leading to spontaneous magnetization.

This method, as pointed out by Lipowski and Suzuki [1], yields the exact critical temperature of the Ising model with nearest-neighbour interaction on many two-dimensional lattices (square, triangular, honeycomb and centered square), and very accurate estimates of the critical temperature of more complicated models (Ising model with alternating strength of interaction, with next-nearest-neighbour interaction, $S \geqslant 1$ models).

In the present section we will show, resorting to a result by Au-Yang and Fisher [4], that at least in the simplest case of the nearest-neighbour Ising model on the square lattice the LS approximation yields not only the exact critical temperature, but also the *exact boundary magnetization* of the semi-infinite model.

Let us consider an Ising model on a strip $\mathcal{S}_n$ of a square lattice, number from 1 to n the chains which form the strip and apply a magnetic field h_n on the nth chain. The corresponding Hamiltonian will be

$$-\beta H = K \sum_{i=-\infty}^{+\infty} \sum_{j=1}^{n-1} (s_{i,j}s_{i+1,j} + s_{i,j}s_{i,j+1}) + K \sum_{i=-\infty}^{+\infty} s_{i,n}s_{i+1,n} + h_n \sum_{i=-\infty}^{+\infty} s_{i,n} \tag{2}$$

where $s_{i,j} = \pm 1$ is an Ising spin located at the site with coordinates i and j in the x and y direction respectively. The magnetization $m_{1n} \equiv m_{1n}(K, h_n) = \langle s_{i,1} \rangle$ has been calculated for $n \geqslant 2$ by Au-Yang and Fisher in [4], and is given by

$$m_{1n}(K, h_n) = z' \left(\frac{\tilde{c}_+}{\tilde{c}_-} \right)^{1/2} \left[\frac{|\tilde{t}| - \tilde{t} \tanh(2n \sinh^{-1}|t'|)}{|\tilde{t}| + (\tilde{t} + \tilde{c}_+ z'^2)\tanh(2n \sinh^{-1}|t'|)} \right]^{1/2} . \tag{3}$$

In (3) we have adopted the same notation as [4], i.e.

$$z' = \tanh h_n$$

$$t' = \frac{1 - \sinh 2K}{(2\sinh 2K)^{1/2}} \qquad \tilde{t} = t'(1 + t'^2)^{1/2} = \tfrac{1}{2}(\coth 2K - \cosh 2K)$$

$$\tilde{c}_\pm = (2 + t'^2)^{1/2}(1 + t'^2)^{1/2} - \tilde{t} \pm 1 = \cosh 2K \pm 1 .$$

For $K > K_c = \frac{1}{2}\ln(1+\sqrt{2})$ it is easily realized that if h_n is chosen in such a way that $\tilde{t} + \tilde{c}_+ z'^2 = -\tilde{t}$, i.e.

$$z' = \tanh h_n = \left(\frac{\cosh 2K - \coth 2K}{\cosh 2K + 1}\right)^{1/2} \tag{4}$$

then the quantity in square brackets in (3) equals 1, independent of n, and one has

$$m_{1n}(K, h_n) \equiv m_1(K) = \left(\frac{\cosh 2K - \coth 2K}{\cosh 2K - 1}\right)^{1/2} \tag{5}$$

where $m_1(K)$ is the *exact boundary magnetization* of the two-dimensional semi-infinite Ising model [5]. Furthermore, for $K < K_c$, choosing $h_n = 0$ yields $m_1(K) = 0$, again independent of n.

The case $n = 1$, for which (3) is not valid, can be easily solved by the transfer matrix method, giving

$$m_{11}(K, h_1) = \frac{\sinh h_1}{\left(\sinh^2 h_1 + e^{-4K}\right)^{1/2}} \tag{6}$$

which again equals $m_1(K)$ if h_1 is chosen according to (4).

These results imply that, no matter which n, n' we choose, the bifurcation at $h_{\text{eff}} = 0$ will always occur at the exact critical point, $K = K_c$. This mechanism is at the basis of the results of [1] for the square lattice case, and we believe that the same should work for the other two dimensional lattices.

Finally, if, fully in the spirit of a classical approach, we consider the non-zero magnetization solution in (1) for $K > K_c$, by putting $h_{\text{eff}} = h_n$ we obtain the exact spontaneous boundary magnetization m_1, independent of n, n'. This was not noticed in [1].

The following remarks are in order for an explanation of the above results. First of all, the boundary magnetization is known to behave as $m_1(K) \approx (K - K_c)^{\beta_1}$ for $K \to K_c^+$ with $\beta_1 = \frac{1}{2}$, in the 2D Ising model. The exponent $\beta_1 = \frac{1}{2}$ [5] is such that it can be reproduced exactly by a self-consistent approach like the LS approximation, making use of an effective field. When considering models with β_1 values incompatible with a classical scheme, one has to consider the LS approach and its possible extensions and approximations, as we will discuss in the next sections.

As shown in [6], in the context of a generalized cluster variation approach to two-dimensional lattice models, the double strip S_2 is able to contain all of the information needed to solve exactly the two-dimensional NN Ising model. The problem then reduces to how such information can be extracted. Clearly what we presented in this section amounts to a relatively simple way of obtaining part of this information.

3. Mean field renormalization group and finite size scaling

Let us now see how the LS approximation can be used to develop a new MFRG strategy, where the boundary magnetization is used together with the bulk one as an effective scaling operator. This will also be useful in understanding the relations of the LS method with FSS.

The notation applies to an Ising model for convenience, but the strategy is not limited to this case, as will be shown in the next section where it will be used to investigate the three-state Potts model.

For infinite Ising strips of widths n and n', FSS implies the following scaling law for the singular part of the bulk free-energy density $f^{(b)}$:

$$f_{n'}^{(b)}(\ell^{y_T}\epsilon, \ell^{y_H}h) = \ell^d f_n^{(b)}(\epsilon, h) \tag{7}$$

where $\epsilon = (T_c - T)/T_c$, $\ell = n/n'$ is the rescaling factor, and d is the bulk dimension. If the boundary conditions are open, for the singular part of the surface free energy density, $f^{(s)}$, the relation

$$f_{n'}^{(s)}(\ell^{y_T}\epsilon, \ell^{y_H}h, \ell^{y_{HS}}b) = \ell^{d-1} f_n^{(s)}(\epsilon, h, b) \tag{8}$$

holds, where b indicates a surface field.

The basic idea of the MFRG [3] is to derive from (7) the scaling relation for the bulk magnetization

$$m_{n'}(K', h') = \ell^{d-y_H} m_n(K, h) \tag{9}$$

where $h' = \ell^{y_H}h$ and $K' \equiv K'(K)$ is a mapping in the Wilson–Kadanoff sense, determined implicitly, in the limit of h going to zero, on the basis of (9). From this mapping the critical point K_c and the thermal exponent y_T can be obtained by means of the relations $K_c = K'(K_c)$ and $\ell^{y_T} = \frac{\partial K'}{\partial K}(K = K_c)$.

Applying this idea to the surface magnetization yields

$$m_{1n'}(K', h', b') = \ell^{d-1-y_{HS}} m_{1n}(K, h, b) \tag{10}$$

where $b' = \ell^{y_{HS}}b$ if we want b to scale as a surface field. On the other hand, the equation for the critical point which is characteristic of the LS approximation would be recovered if b scaled as a magnetization, i.e. with an exponent $d - 1 - y_{HS}$. In fact, with this assumption, setting $h = 0$ and linearizing in b yields

$$\frac{\partial m_{1n'}}{\partial b'}(K', 0, 0) = \frac{\partial m_{1n}}{\partial b}(K, 0, 0) \tag{11}$$

which implicitly defines a mapping $K'(K)$. The equation

$$K_c = K'(K_c) \tag{12}$$

with $K'(K)$ given by (11) is equivalent to the equation for the critical point in the LS approximation. So this approximation can also be seen as a realization of a MFRG strategy as far as determination of K_c is concerned.

In a MFRG spirit one can also determine the critical exponents, since y_T is obtained by the relation

$$\ell^{y_T} = \left.\frac{\partial K'}{\partial K}\right|_{K=K_c}. \tag{13}$$

Linearizing (9) with respect to h, with $K = K_c$ yields

$$\frac{\partial m_{n'}}{\partial h'}(K_c, 0) = \ell^{d-2y_H} \frac{\partial m_n}{\partial h}(K_c, 0) \tag{14}$$

from which y_H can be obtained and finally, linearizing (10) in the same way, with $b = 0$, yields

$$\frac{\partial m_{1n'}}{\partial h'}(K_c, 0, 0) = \ell^{d-1-y_{HS}-y_H} \frac{\partial m_{1n}}{\partial h}(K_c, 0, 0) \tag{15}$$

from which y_{HS} is obtained.

The set of equations (11)–(15) is a MFRG procedure to determine critical point and critical exponents.

Nevertheless, the procedure above (to be denoted by M, for 'magnetization', in the following) is not a rigorous application of FSS. In such an application (11) should be replaced by

$$\frac{\partial m_{1n'}}{\partial b'}(K', 0, 0) = \ell^{d-1-2y_{HS}} \frac{\partial m_{1n}}{\partial b}(K, 0, 0) \tag{16}$$

since b should scale as a field, with exponent y_{HS}, and should be solved in conjunction with (14)–(15). This alternative and more rigorous procedure will be denoted by F, for 'field', in the following.

F is in fact the procedure of MFRG proposed in [3] to yield simultaneously bulk and surface exponents. It is interesting to investigate how M, proposed here, being more consistent with the effective field idea, compares with F.

Two comments are in order:

(i) The two procedures should give the same value of K_c (but not of the exponents) in the limit $n, n' \to \infty$, $\ell \to 1$, since the two derivatives in (11) are analytic functions; this should justify the LS approximation in a FSS context.

(ii) In the two-dimensional NN Ising case, $y_{HS} = \frac{1}{2}$ exactly and then, in the limit above, the critical exponents should also be the same for both procedures M and F.

In the following section we will check these ideas on the two-dimensional NN Ising and three-state Potts cases..

4. Results and discussion

In the present section we give two test applications of our new MFRG strategy (M), to the Ising and three-state Potts models on square lattices. We also compare our results with those obtained treating b as a surface field, i.e. letting it scale with exponent y_{HS}. We start by applying procedure M to the Ising model. In the Ising case, we have already shown that the method gives the exact critical point for any n, n'. Furthermore, resorting to (3), the mapping $K' = K'(K)$ can be determined analytically in an implicit form. As a result one gets

$$f_{n'}(K') = f_n(K) \tag{17}$$

where

$$f_n(K) = K \coth K \left[\frac{1 + \tanh(2n \sinh^{-1}|t'|)}{1 - \tanh(2n \sinh^{-1}|t'|)}\right]^{1/2} \tag{18}$$

and with t' as above. It can be checked that the fixed point of (17), obtained by setting $t' = 0$, is $K^* = \frac{1}{2}\ln(1+\sqrt{2})$, while for the thermal exponent one has

$$\ell^{y_T} = \frac{1 + 2(2n-1)K^*}{1 + 2(2n'-1)K^*} \tag{19}$$

Table 1. Results for the Ising model (b = magnetization).

n	y_T	y_H	y_{HS}
2	0.957 38	1.602 87	0.602 87
3	0.973 10	1.680 92	0.568 08
4	0.980 89	1.721 97	0.550 99
5	0.985 14	1.747 93	0.540 67
6	0.987 85	1.766 00	0.533 74
7	0.989 69	1.779 37	0.528 73
8	0.991 06	1.789 70	0.524 96
Extrapolated	0.999 77	1.872 01	0.497 64
Exact	1	1.875	0.5

which, in the limit $n, n' \to \infty$, yields $y_T = 1$, which is again an exact result.

The calculation of the magnetic exponents cannot be carried out analytically since no solution is available for the bulk magnetization of a strip in the presence of a bulk magnetic field, and we have to proceed numerically as follows: given the strip S_n with a bulk magnetic field h and an auxiliary magnetic field h_1 acting on the first chain, we determine its partition function $Z_n(K, h, h_1)$ as the largest eigenvalue of the $2^n \times 2^n$ transfer matrix with elements

$$T_n(\{s_j\}, \{s_j'\}) = \exp\left[\frac{K}{2}\sum_{j=1}^{n-1}(s_j s_{j+1} + s_j' s_{j+1}') + K\sum_{j=1}^{n} s_j s_j' + \frac{h}{2}\sum_{j=1}^{n}(s_j + s_j') + \frac{h_1}{2}(s_1 + s_1')\right]. \tag{20}$$

The bulk and boundary magnetizations will then be given by

$$m_n \equiv m_n(K, h) = \left.\frac{1}{nZ_n}\frac{\partial Z_n}{\partial h}\right|_{h_1=0} \tag{21}$$

and

$$m_{1n} \equiv m_{1n}(K, h) = \left.\frac{1}{Z_n}\frac{\partial Z_n}{\partial h_1}\right|_{h_1=0} \tag{22}$$

respectively. Finally, y_H and y_{HS} are determined according to (14)–(15).

In table 1 we report the results for the critical exponents for strip widths $2 \leqslant n = n'+1 \leqslant 8$ ($n = n' + 1$ is always the most convenient choice). The extrapolations are based on least-squares fits with fourth-order polynomials in $1/n$ and are certainly justified since the critical exponents are nearly linear functions of $1/n$. The agreement of the extrapolated data with the exact results is very good, and the errors are within 0.2%.

The Ising test has yielded very promising results, but does not shed much light on the physical meaning of the effective parameter b, since for the two-dimensional Ising model surface magnetization and surface field scale with the same exponent $d - 1 - y_{HS} = y_{HS} = \frac{1}{2}$. In fact, procedure F, in which b scales as a surface field, also yields very good results (apart from having no solution when $n = 2$), as shown in table 2. Here the extrapolations are based on the so-called alternating ϵ-algorithm [2, 11]. Data from the two procedures are plotted together in figures 1–4. As expected, all the results seem to be equivalent in the limit $n \to \infty$.

In view of the above considerations, we believe that a more conclusive test is in order, and a suitable model should be the three-state Potts model. Indeed, in two dimensions, this model is known to undergo a second-order phase transition, whose critical point and critical

Table 2. Results for the Ising model (b = field).

n	K_c	y_T	y_H	y_{HS}
3	0.529 03	0.965 48	1.990 12	0.911 50
4	0.480 65	0.978 40	1.911 95	0.770 40
5	0.463 66	0.984 04	1.885 70	0.702 67
6	0.455 60	0.987 26	1.873 97	0.661 95
7	0.451 13	0.989 38	1.867 94	0.634 55
8	0.448 39	0.990 86	1.864 61	0.614 82
Extrapolated	0.440 87	1.006 20	1.858 53	0.496 81
Exact	0.440 69	1	1.875	0.5

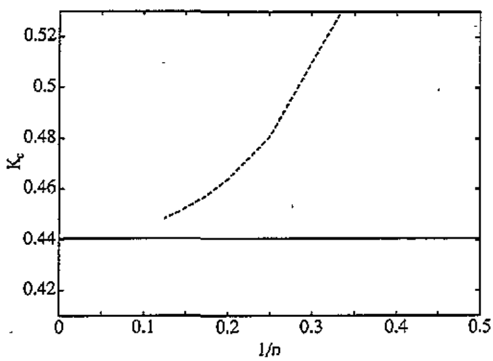

Figure 1. The Ising critical point K_c versus $1/n$ as given by the M (full curve) and F (broken curve) procedures.

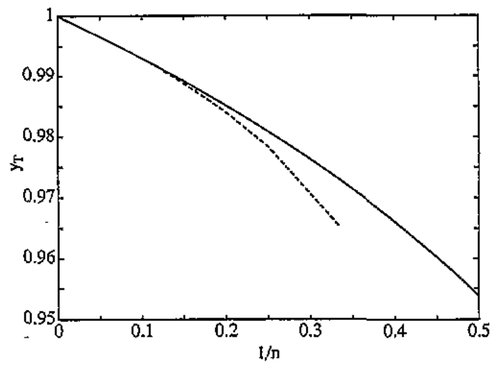

Figure 2. The The Ising thermal exponent y_T versus $1/n$ as given by the M (full curve) and F (broken curve) procedures.

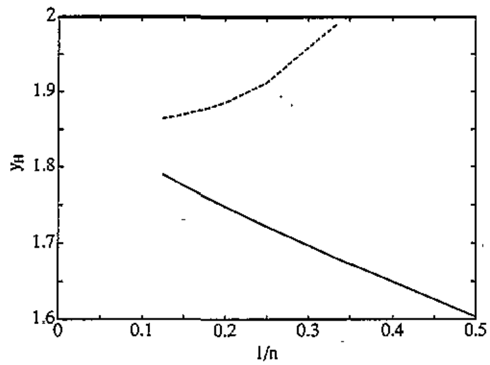

Figure 3. The Ising magnetic exponent y_H versus $1/n$ as given by the M (full curve) and F (broken curve) procedures.

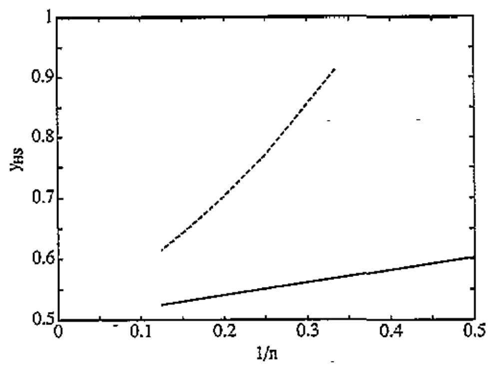

Figure 4. The Ising surface magnetic exponent y_{HS} versus $1/n$ as given by the M (full curve) and F (broken curve) procedures.

exponents are known exactly [7–10], even in the absence of a full solution. In particular it has $y_{HS} \neq \frac{1}{2}$.

The Hamiltonian of the q-state Potts model [7] is

$$-\beta H = \frac{K}{q-1}\sum_{\langle ij\rangle}(q\delta_{s_i,s_j} - 1) + \frac{h}{q-1}\sum_i(q\delta_{s_i,0} - 1) \tag{23}$$

where the variables s_i take on values $0, 1, \ldots q-1$, $K > 0$ is the interaction strength and h is

Table 3. Results for the three-state Potts model (b = magnetization).

n	K_c	y_T	y_H	y_{HS}
2	0.713 18	1.101 52	1.609 16	0.609 16
3	0.697 62	1.121 18	1.690 79	0.569 66
4	0.690 13	1.132 09	1.736 52	0.547 85
5	0.685 74	1.139 78	1.767 04	0.533 50
6	0.682 87	1.144 96	1.789 23	0.523 17
7	0.680 84	1.149 93	1.806 29	0.515 26
8	0.679 35	1.151 59	1.819 91	0.509 11
Extrapolated	0.669 01	1.174 48	1.912 10	0.469 98
Exact	0.670 04	1.2	1.866 67	0.333 33

Table 4. Results for the three-state Potts model (b = field).

n	K_c	y_T	y_H	y_{HS}
3	0.789 99	1.128 71	1.965 88	0.873 40
4	0.726 12	1.140 60	1.887 67	0.720 35
5	0.703 02	1.146 27	1.861 66	0.642 90
6	0.691 84	1.150 25	1.850 30	0.594 17
7	0.685 55	1.155 41	1.844 77	0.560 18
8	0.681 65	1.157 31	1.841 94	0.534 83
Extrapolated	0.670 72	1.155 36	1.838 37	0.365 87
Exact	0.670 04	1.2	1.866 67	0.333 33

a magnetic field. The order parameter of the model, corresponding to the Ising magnetization, is

$$m = \frac{q\langle\delta_{s_i,0}\rangle - 1}{q-1}. \tag{24}$$

In the case $q = 2$ one recovers the Ising model.

The MFRG scheme developed above can be carried over to the q-state Potts model without any substantial modification, and we will apply it to the case $q = 3$. The main new fact is that no analytical results like (3) are available for the three-state Potts model. So all calculations must be performed numerically with the transfer matrix method. The order of the transfer matrix is now 3^n and increases more rapidly than in the Ising case. However, the transfer matrix is invariant with respect to the transformation which interchanges the states $s_i = 1$ and $s_i = 2$ (all other symmetries are lost as soon as one introduces the surface fields), and the eigenvector corresponding to its largest eigenvalue belongs to the symmetric subspace of this transformation. Thus we can limit ourselves to matrices acting in this subspace, which are of order $(3^n + 1)/2$. In this way we have been able to deal with strips up to $n = 8$.

The numerical results for procedure M are reported in table 3. The extrapolation is obtained by fitting data in a least-square sense with a second-order polynomial in $1/n$. Even if now the critical point is not given exactly by the LS approximation, we obtain excellent agreement with the exact results (errors within 2.5%) for the bulk critical point and exponents, while, in comparison, the results for the surface exponent y_{HS} are rather poor.

In table 4 we report the results obtained from procedure F. The situation is different from the previous one: the errors of the extrapolated critical point and bulk exponents are within 3.8% and the error for the surface exponent is within 10%. In particular, the estimate for the surface critical exponent is comparable with the value 0.343 obtained by phenomenological renormalization in [12]. Results from the two procedures are plotted together in figures 5–8.

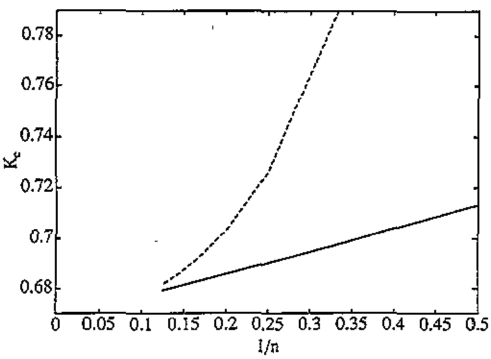

Figure 5. The three-state Potts critical point K_c versus $1/n$ as given by the M (full curve) and F (broken curve) procedures.

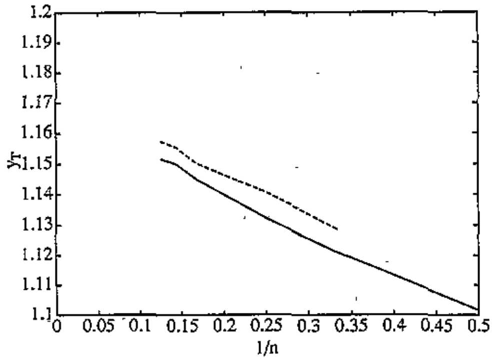

Figure 6. The three-state Potts thermal exponent y_T versus $1/n$ as given by the M (full curve) and F (broken curve) procedures.

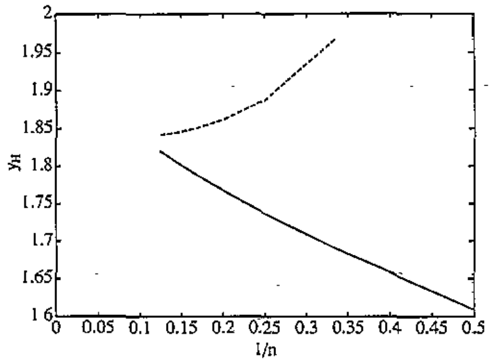

Figure 7. The three-state Potts magnetic exponent y_H versus $1/n$ as given by the M (full curve) and F (broken curve) procedures.

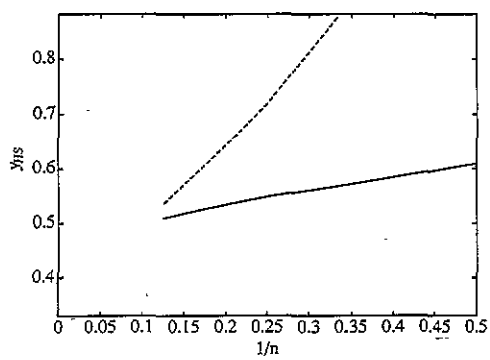

Figure 8. The three-state Potts surface magnetic exponent y_{HS} versus $1/n$ as given by the M (full curve) and F (broken curve) procedures.

There is strong evidence that F is the correct procedure when n is large, but for small strips M, although not rigorous, seems to work very well: indeed, if y_H had been extrapolated on the basis of the results for $2 \leqslant n \leqslant 5$ one would have obtained 1.868, which is two orders of magnitude more accurate than the extrapolation on the whole set of data.

5. Conclusions

We have analysed in some detail the LS approximation, showing that in the two-dimensional NN Ising case, it yields not only the exact critical point, but also the exact boundary magnetization of the semi-infinite model, independent of the size of the strips used. We have also proposed an explanation of these surprising results.

The LS approximation has been used to develop a new MFRG strategy (procedure M) which yielded very accurate results for the critical exponents of the Ising and three-state Potts models in two dimensions. When compared with rigorous FSS (procedure F) our new strategy has proven to be particularly suitable for applications where relatively small strips are used, while for larger strips our results indicate quite clearly that F is the correct procedure.

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